UDC 62-50

ON A PROBLEM OF CORRECTION OF THE CONTROLLED PROCESS

PMM Vol. 42, No. 6, 1978, pp. 1127-1131

L. G. VASIL'EVA and V. N. LAGUNOV
(Kalinin)

(Received July 1, 1977)

The results obtained in [1] are developed further, by considering a case in which the processs is described by a nonlinear, differential vector equation of the p -th order, the correction vector appears in the right hand side of the equation of motion as one of the terms, and the noise vector is a measurable vector function. A constructive method of correcting the process in question is given, with the correction equation obtained in the explicit form.

1. Let the course of some process be described by the following nonlinear differential vector equation:

$$z^{(p)} = f(z, z^{\cdot}, \ldots, z^{(p-1)}, u^2) + u^1(t) \tag{1.1}$$
$$z^{(i)}(t_0) = z_0^{(i)}, \quad i = 0, \ldots, p-1, \quad t \in [t_0, t']$$

Here $z \in E_n$, $z^{(i)}$ $(i = 1, \ldots, p)$ are the i -th order time derivatives of the vector z , $u^1(t)$ is a piecewise constant vector function (correction), $u^2(t)$ is a measurable vector function (noise) and $f(z, z^{\cdot}, \ldots, z^{(p-1)}, u^2)$ is a vector function continuous in all arguments in question and satisfying the condition

$$| f(z, z^{\cdot}, \ldots, z^{(p-1)}, u^2) | \leqslant M, \quad M = \text{const} \tag{1.2}$$

We introduce the vector y with components $y_i = z^{(i-1)}$ $(i = 1, 2, \ldots, p)$ and vector function $\varphi'(y, u^1, u^2)$ the components of which have the form

$$\varphi_i' = y_{i+1}, \quad i = 1, 2, \ldots, p-1$$
$$\varphi_p' = f(y_1, y_2, \ldots, y_p, u^2) + u^1(t)$$

Then the equation (1.1) can be written in the form

$$y_1^{\cdot} = y_2, \quad y_2^{\cdot} = y_3, \ldots, \quad y_{p-1} = y_p \tag{1.3}$$
$$y_p^{\cdot} = f(y_1, \ldots, y_p, u^2) + u^1(t)$$

From (1.2), the continuity of $f(y_1, y_2, \ldots, y_p, u^2)$ in y_i $(i = 1, \ldots, p)$, u^2 and the measurability of $u^2(t)$ follows the existence and uniqueness of the solution of the system (1.3) in the sense of Carathèodory [2], in the form of absolutely continuous vector functions of time $y_1(t), y_2(t), \ldots, y_p(t)$ satisfying the conditions

$$y_i(t_0) = y_{i,0}, \quad i = 1, \ldots, p$$

Let x be a solution of the system (1.3) with $u^1(t) \equiv u^2(t) \equiv 0$. In this case we obtain the following system of equations:

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = x_3, \ldots, \quad \dot{x}_{p-1} = x_p \tag{1.4}$$
$$x_p^{\cdot} = f(x_1, x_2, \ldots, x_p, 0)$$

and initial conditions

$$x_i(t_0) = y_i(t_0), \quad i = 1, \ldots, p$$

As in [1], we shall treat the problem of correction as a differential game in which the functions u^i described above are the admissible controls of the i-th player. The admissible strategies of the first player will consist of the piecewise programmable strategies placing the quantities t_k, $x_i(t_k)$ and $y_i(t_k)$, $i = 1, \ldots, p$, in 1:1 correspondence with the number $t_{k+1} = t_k + \Delta t_k$ and the vector function $u_k^1(t)$, $t \in [t_k, t_k + \Delta t_k]$, the latter representing a restricted form of some admissible control of the first player on $[t_k, t_k + \Delta t_k] \subset [t_0, t']$. Any strategy is admissible for the second player provided it satisfies the following single requirement: that the control formulated with its help must be admissible. We shall assume that the game under consideration discriminates the first player.

Let us denote by v^i the admissible strategy of the i-th player, and by V^i the class of his admissible strategies. The payoff in the game in question is defined by

$$J(v^1, v^2) = \sup_{t \in [t_0, t']} |x_1(t) - y_1(t)|$$

where $x_1(t)$ and $y_1(t)$ are absolutely continuous vector functions of time appearing in the systems (1.3) and (1.4).

Let us pose the following problem: to describe constructively the strategy $v_0^1 \in V^1$ for which the following inequality holds:

$$J(v_0^1, v^2) \leqslant \varepsilon, \quad \forall v^2 \in V^2 \tag{1.5}$$

2. Let us consider a representation of the function $x_1(t)$ given in the form of an analog of the Taylor's formula [3], in which the remainder term is given in the integral form. By virtue of (1.4) we have, on $[t_k, t_k + \Delta t_k]$

$$x_1(t_k + \Delta t_k) = x_1(t_k) + \sum_{i=2}^{p} x_i(t_k) \frac{\Delta t_k^{i-1}}{(i-1)!} + \int_0^{\Delta t_k} \frac{(\Delta t_k - \xi)^{p-1}}{(p-1)!} f(x_1(t_k + \xi), \ldots, x_p(t_k + \xi), 0)\, d\xi$$

Similarly, the expansion of the function $y_1(t)$ on $[t_k, t_k + \Delta t_k]$ has the form

$$y_1(t_k + \Delta t_k) = y_1(t_k) + \int_0^{\Delta t_k} u^1(t_k + \xi) \frac{(\Delta t_k - \xi)^{p-1}}{(p-1)!}\, d\xi + A(\Delta t_k) \tag{2.1}$$

$$A(\Delta t_k) = \sum_{i=2}^{p} y_i(t_k) \frac{\Delta t_k^{i-1}}{(i-1)!} + \int_0^{\Delta t_k} f(y_1(t_k + \xi), \ldots, y_p(t_k + \xi), u^2(t_k + \xi)) \frac{(\Delta t_k - \xi)^{p-1}}{(p-1)!}\, d\xi$$

Let

$$u^1 (t_k + \xi) = u_k^1 (t_k), \; \xi \in (t_k, t_{k+1}) \tag{2.2}$$

Taking into account (2.2), we can write the expression (2.1) in the following form:

$$y_1 (t_k + \Delta t_k) = y_1 (t_k) + u_k^1 (t_k) \Delta t_k^p/p! + A (\Delta t_k) \tag{2.3}$$

Let us introduce the notation

$$R_1 (t_k, \Delta t_k) = \sum_{i=2}^{p} | y_i (t_k) | \frac{\Delta t_k^{i-1}}{(i-1)!} + M \frac{\Delta t_k^{\,p}}{p!} \tag{2.4}$$

$$R (t_k, \Delta t_k) = \sum_{i=2}^{p} | x_i (t_k) | \frac{\Delta t_k^{i-1}}{(i-1)!} + M \frac{\Delta t_k^{\,p}}{p!} \tag{2.5}$$

From (1.2) follows

$$| A (\Delta t_k) | \leqslant R_1 (t_k, \Delta t_k)$$

Let us formulate a lemma concerning the extrapolation [1] which can be applied to the present problem.

L e m m a 2 . 1. The point $y_1 (t_k)$ will be transported under the action of the control (2.2) and any control $u^2 (t)$, in time Δt_k, to the point $y_1 (t_k + \Delta t_k)$ lying within a sphere with its center at the point

$$y_1 (t_k) + u_k^1 (t_k) \Delta t_k^p / p!$$

(see (2.3)), of radius $R_1 (t_k, \Delta t_k)$ (2.4).

If we denote by $S (c_0, R)$ a closed sphere in E_n with the center at the point c_0, of radius R, then we can conclude the lemma as follows:

$$y_1 (t_k + \Delta t) \in S (y_1 (t_k) + u_k^1 (t_k) \Delta t_k^p / p!, \; R_1 (t_k, \Delta t_k))$$

Let us consider a particular case of Lemma 2.1 in which the controls $u^1 (t)$ and $u^2 (t)$ are both zero. We find that by the time $t_k + \Delta t_k$ the point $x_1 (t_k)$ will be transported to the point $x_1 (t_k + \Delta t_k)$ and

$$x_1 (t_k + \Delta t_k) \in S (x_1 (t_k), \; R (t_k, \Delta t_k))$$

The validity of the lemma and the latter assertion follows directly from the relations (2.3) – (2.5).

3. Let us introduce the relation

$$r (t_k + \tau) = | x_1 (t_k + \tau) - y_1 (t_k + \tau) | \tag{3.1}$$

From (3.1) we obtain the inequality

$$\begin{aligned} r (t_k + \Delta t_k) \leqslant r_1 (t_k + \Delta t_k) \equiv | [y_1 (t_k) - x_1 (t_k)] + \\ u_k^1 (t_k) \Delta t_k^p / p! | + R (t_k, \Delta t_k) + R_1 (t_k, \Delta t_k) \end{aligned} \tag{3.2}$$

Consider a system of two equations with two unknowns u_k^1 and Δt, which follow from (3.2) and (1.5):

$$\begin{aligned} y_1 (t_k) - x_1 (t_k) + u_k^1 (t_k) \Delta t^p / p! = 0 \\ R (t_k, \Delta t) + R_1 (t_k, \Delta t) = \varepsilon / 2 \end{aligned} \tag{3.3}$$

The second equation of (3.3) has a unique positive root Δt_k. In this case the control u_k^1 is given uniquely by the first equation of the above system, in the form

$$u_k^1 = (x_1(t_k) - y_1(t_k))p! \,/\, \Delta t_k^p \tag{3.4}$$

T h e o r e m 3 . 1. If at some instant of time t_k

$$r(t_k) \leqslant \varepsilon / 2 \tag{3.5}$$

and u_k^1 , Δt_k satisfy the system of equations (3.3), then the following inequality holds within the time interval $[t_k, t_k + \tau]$, $0 < \tau \leqslant \Delta t_k$:

$$r(t_k + \tau) \leqslant \varepsilon,\ r(t_k + \Delta t_k) \leqslant \varepsilon / 2,\ \tau \in (0, \Delta t_k] \tag{3.6}$$

P r o o f. The relations (3.2), (3.3) and (3.5) yield the second inequality of (3.6). Since the function $R(t_k, \tau) + R_1(t_k, \tau)$ is an increasing function of τ , we have for $\tau \in (0, \Delta t_k]$

$$R(t_k, \tau) + R_1(t_k, \tau) \leqslant R(t_k, \Delta t_k) + R_1(t_k, \Delta t_k) = \varepsilon / 2$$

The method of obtaining the control u_k^1 (see (3.4)) on $[t_k, t_k + \tau]$, $\tau \in (0, \Delta t_k]$, implies the inequality

$$|[y_1(t_k) - x_1(t_k)] + u_k^1(t_k)\tau^p / p!| \leqslant |y_1(t_k) - x_1(t_k)| \leqslant \varepsilon / 2$$

Replacing Δt_k by τ in the last two relations and (3.2), we find that the first inequality of (3.6) holds.

We have the following estimate for the absolute value of the control u_k^1 on $[t_k, t_k + \Delta t_k]$:

$$|u_k^1(t_k)| \leqslant (|x_1(t_k) - y_1(t_k)|)\, p! \,/\, \Delta t_k^p \leqslant \varepsilon p! \,/\, (2\Delta t_k^p)$$

4. Let us generalize the relations obtained to the case when the correction control has the form

$$u^1(t) = \begin{cases} 0, & t \in [t_k, t_k + \Delta\tau],\ \Delta\tau > 0 \\ u_*^1 = \text{const}, & t \in [t_k + \Delta\tau, t_k + \Delta\tau + \tau],\ \tau > 0 \end{cases} \tag{4.1}$$

From the lemma on extrapolation and the form of (4.1), we obtain

$$y_1(t_k + \theta) \in S(y_1(t_k),\ R'(t_k, \theta)), \qquad \theta \in [0, \Delta\tau] \tag{4.2}$$

$$R'(t_k, \theta) = \sum_{i=2}^{p} \frac{\theta^{i-1}}{(i-1)!}|y_i(t_k)| + M\frac{\theta^p}{p!}$$

and (4.1), (4.2) together yield the inclusion

$$y_1(t_k + \Delta\tau + \varphi) \in S\left(y_1(t_k) + u_*^1\frac{\varphi^p}{p!},\ R'(t_k, \Delta\tau) + R_1(t_k + \Delta\tau, \varphi)\right) \tag{4.3}$$

$\varphi \in [0, \tau]$

We have the following relation for (4.1):

$$r(t_k + \theta) \leqslant r_1(t_k + \theta) \equiv |y_1(t_k) - x_1(t_k)| + R(t_k, \theta) + R'(t_k, \theta), \quad \theta \in [0, \Delta\tau] \tag{4.4}$$

Relations (4.3) and (2.5) together imply the validity of the inequality

$$r(t_k + \Delta\tau + \varphi) \leqslant r_2(t_k + \Delta\tau + \varphi) \equiv |y_1(t_k) - x_1(t_k) + u_*^1\varphi^p / p!| + R(t_k, \Delta\tau + \varphi) + R'(t_k, \Delta\tau) + R_1(t_k + \Delta\tau, \varphi) \tag{4.5}$$

Let the numbers α and β satisfy the following conditions:

$$0<\alpha<1, 0<\beta<1, \alpha+2\beta<1 \tag{4.6}$$

Lemma 4.1. If

$$r(t_k) \leqslant \varepsilon\alpha / 2, \quad u^1(t)=0, \quad t \geqslant t_k \tag{4.7}$$

then

$$r(t_k+\tau) \leqslant \varepsilon(1-\beta)/2, \quad \tau \in [0, \tau_\alpha]$$

where τ_α is the root of the equation

$$R(t_k, \tau)+R'(t_k, \tau)=\varepsilon(1-\beta-\alpha)/2 \tag{4.8}$$

Proof. From (4.1) and (4.7) follows the validity of the inequality

$$r(t_k+\tau) \leqslant \varepsilon\alpha/2+R(t_k, \tau)+R'(t_k, \tau) \tag{4.9}$$

The solution τ_α of (4.8) exists and is unique (the function in the left hand side of (4.9) vanishes when $\tau=0$ and increases without bounds as $\tau \to \infty$). Replacing in (4.9) τ by τ_α, we obtain the result of the lemma.

Lemma 4.2. If

$$r_1(t_k) \leqslant \varepsilon(1-\beta)/2, \ u^1(t)=0, \ t \geqslant t_k$$

then

$$r_1(t_k+\tau) \leqslant \varepsilon/2, \ \tau \in (0, \tau_\beta]$$

where τ_β is the root of the equation

$$R(t_k, \tau)+R'(t_k, \tau)=\varepsilon\beta/2 \tag{4.10}$$

The condition of the lemma together with (4.4) yields the inequality analogous to (4.9)

$$r_1(t_k+\tau) \leqslant \varepsilon(1-\beta)/2+R(t_k, \tau)+R'(t_k, \tau)$$

and this, together with (4.10), completes the proof of the validity of the lemma.

For τ_α, τ_β, R and R' we have

$$\tau_\alpha>\tau_\beta, \quad R(t_k, \tau_\beta)+R'(t_k, \tau_\beta) \leqslant \varepsilon\beta/2,$$

Lemma 4.3. If

$$r(t_k) \in (\varepsilon\alpha/2, \ \varepsilon(1-\beta)/2)$$

and u_k^1, τ_k satisfy the system of equations

$$\begin{gathered} y_1(t_k)-x_1(t_k)+u_k^1\tau_k^p/p!=0 \\ R(t_k, \tau_\beta+\tau_k)+R_1(t_k+\tau_\beta, \tau_k)+R'(t_k, \tau_\beta)=\varepsilon(1-\beta)/2 \end{gathered} \tag{4.11}$$

then, for the control

$$u_k^1(t)=\begin{cases}0, & t \in [t_k, t_k+\tau_\beta) \\ u_k^1, & t \in [t_k+\tau_\beta, t_k+\tau_\beta+\tau_k]\end{cases}$$

the following relations hold:

$$r_1(t_k+\tau) \leqslant \varepsilon/2, \ r(t_k+\tau) \leqslant \varepsilon/2, \ \tau \in (0, \tau_\beta] \tag{4.12}$$

$$r_2(t_k+\tau_\beta+\tau_k) \leqslant \varepsilon(1-\beta)/2, \quad r_2(t_k+\tau_\beta+\tau) \leqslant \varepsilon(1-\beta), \quad \tau \in (0, \tau_k) \tag{4.13}$$

Proof. Let τ_β be the root of (4.10). Then τ_β always exists and $\tau_\beta>0$ (for the same reasons as τ_α). By Lemma 4.2 the first inequality of (4.12) holds on

$[t_k, t_k + \tau]$, $\tau \in (0, \tau_\beta]$ and (4.4) implies the validity of the second inequlity.

Analyzing the system (4.11) we see that its solution exists and is unique. The second equation of the system depends only on τ_k. Solving this equation we obtain τ_k and substituting the latter into the first equation, we find u_k^1. The method of determining τ_k (see second equation in (4.11)) and the relations (4.5) together imply the validity of the first inequality of (4.13). The proof of the second inequality follows that of Theorem 3.1.

Let us now describe the algorithm for the solution of the problem in question. We choose the numbers α and β which satisfy the conditions (4.6). Then we determine the strategy v_0^1 as follows: we find the numbers τ_α and τ_β; we assume that $u_k^1(t) = 0$ at the instant t_k, and we obtain $r(t_k)$; if $r(t_k) \leqslant \varepsilon\alpha / 2$, then we set $u_k^1(t) = 0$, $t \in [t_k, t_k + \tau_\alpha]$; if $r(t_k) \in (\varepsilon\alpha, \varepsilon(1 - \beta) / 2)$, then we solve the system (4.11) and this yields the control u_k^1, the number τ_k and the supplementary definition of the control

$$u_k^1(t) = u_k^1, \qquad t \in [t_k + \tau_\beta, t_k + \tau_\beta + \tau_k]$$
$$t_k + \tau_\beta + \tau_k = t_{k+1}$$

The validity of (1.5) is secured by applying the strategy v_0^1 at each step.

REFERENCES

1. Lagunov, V. N. On correction of a nonlinear controlled process. PMM Vol. 41, No. 2, 1977.

2. Sansone, G. Equazioni Differenziali di Campo Reale. 2-do ed., Bologna, N. Zanichelli, 1948 —9.

3. Cartan, H. Differential Calculus, Paris Hernan, Boston (Mass.), Houghton Mifflin, 1971.

Translated by L. K.